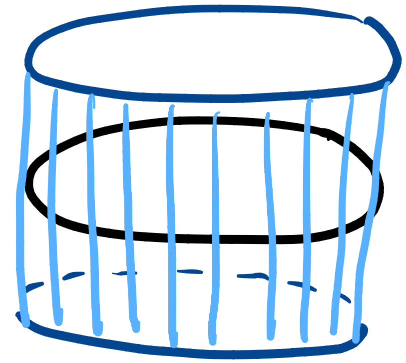
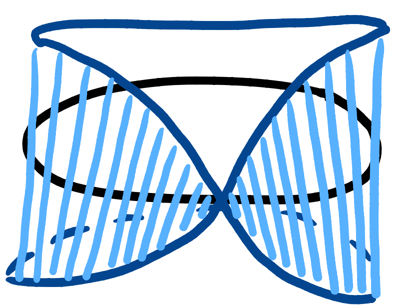
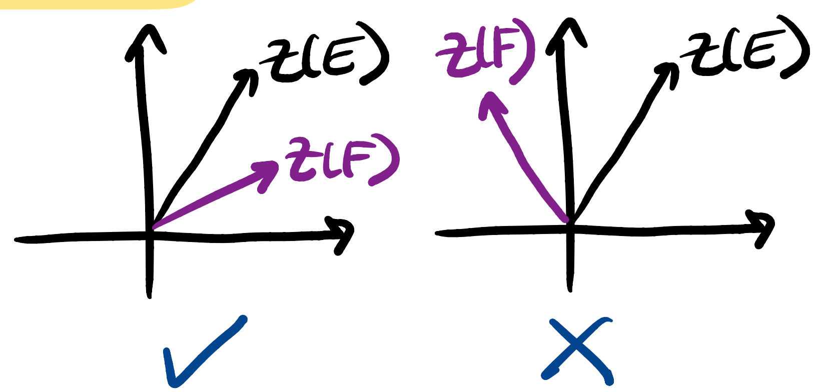
$$\begin{tikzcd} G \arrow[r, phantom, "\overset{\text{free}}{\curvearrowright}"] & X \arrow[r] \arrow[d, "\pi"] & \mathrm{Alb}(X) \\ & S := X/G \arrow[r] & \mathrm{Alb}(S) \end{tikzcd}$$

finite abelian

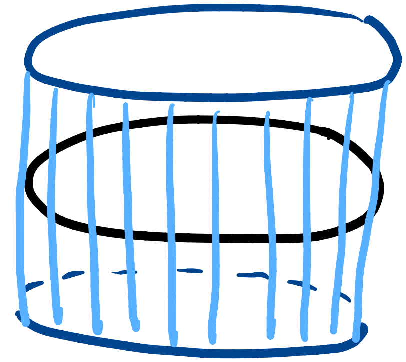


STABILITY CONDITIONS ON FREE ABELIAN QUOTIENTS





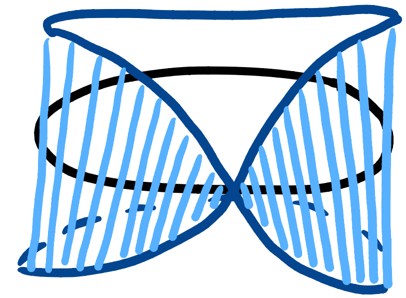
§1 Mumford's Slope Stability



C: smooth projective curve / $\mathbb{C}$

E: vector bundle on C

$\neq 0$



⇝ slope: $\mu(E) := \dfrac{\deg(E)}{\mathrm{rk}(E)}$

Def'n E is μ (semi)-stable if:

$$0 \neq F \subsetneq E \implies \mu(F) \underset{(-)}{<} \mu(E)$$

UPSHOT: Harder-Narasimhan (HN) filtrations

Th'm [Mumford '62] $M^{st}_{r,d}(C)$ is a smooth quasi-proj. variety

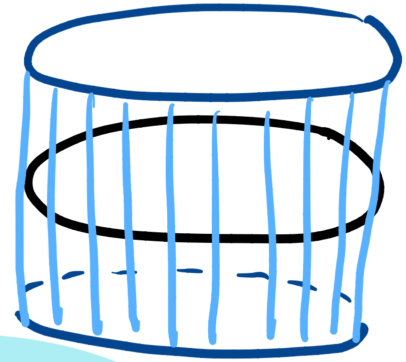
(moduli of μ-stable bundles of rank $= r$, degree $= d$

§1 Mumford's Slope Stability

C: smooth projective curve / $\mathbb{C}$

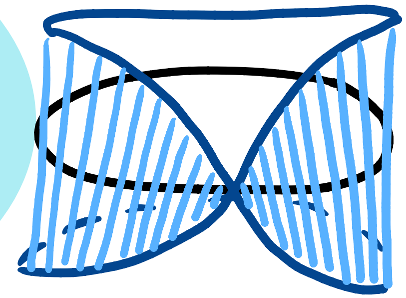
E: ~~vector bundle on C~~ $\in \mathrm{Coh}(C)$

$\neq 0$



$\rightsquigarrow$ slope :

$$\mu(E) := \begin{cases} +\infty & \mathrm{rk}(E) = 0 \\ \dfrac{\deg(E)}{\mathrm{rk}(E)} & \text{o/w} \end{cases}$$



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§2 Bridgeland Stability Conditions (B.S.C.s)

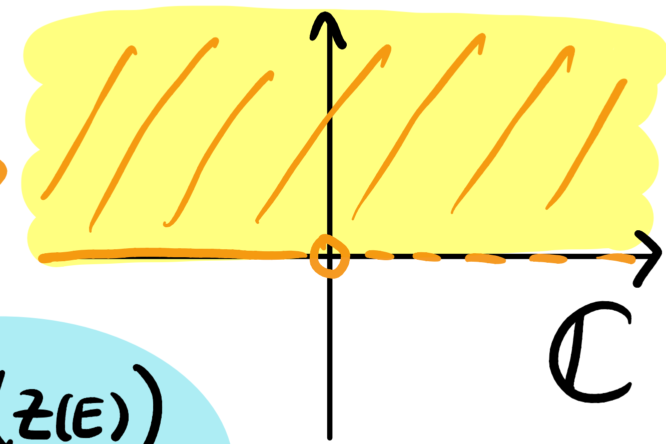
$\mathcal{D}$: triangulated category [e.g. $\mathcal{D} = D^b(X) := D^b(\mathrm{Coh}(X))$]

$E \in \mathcal{A}$ abelian subcat. which "generates" $\mathcal{D}$

$E \neq 0$

$Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ homomorphism

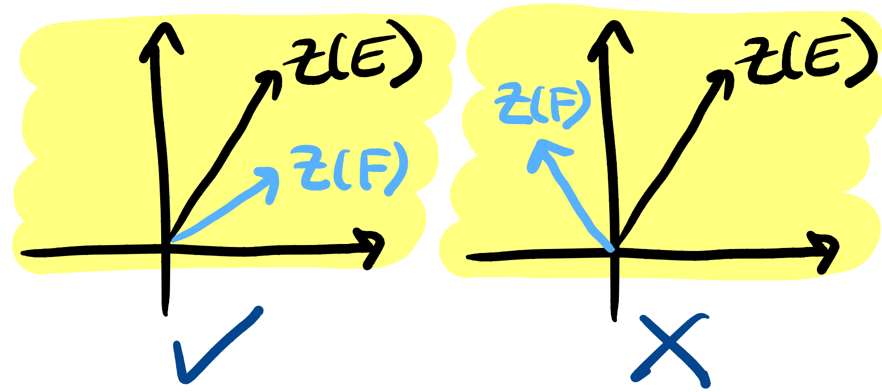
s.t. $Z(E) \in \mathbb{H}$



$\sigma := (\mathcal{A}, Z) \rightsquigarrow \mu_\sigma(E) := \dfrac{-\mathrm{Re}(Z(E))}{\mathrm{Im}(Z(E))}$

E is σ (semi)-stable if:

$\underbrace{0 \neq F \subsetneq E}_{\text{in } \mathcal{A}} \Rightarrow \mu_\sigma(F) \underset{(=)}{<} \mu_\sigma(E)$



σ is a B.S.C. if HN filtrations exist (+ some technical assumptions)

§2 cont.

Def$^{\underline{n}}$ A **Bridgeland stability condition** on $\mathcal{D}$ is a pair
$\sigma = (\mathcal{A}, \mathcal{Z})$: (1) $\mathcal{A} \subseteq \mathcal{D}$ abelian & "generates"
(2) $\mathcal{Z} : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ homomorphism

EXAMPLE $\mathcal{D} = D^b(C)$

$$\sigma_\mu := (\operatorname{Coh}(C), -\deg(E) + i \operatorname{rank}(E))$$

⚠ $E \in \operatorname{Coh}(C)$ is σ_μ-stable $\Leftrightarrow$ μ-stable

$\operatorname{Stab}(\mathcal{D})$:= set of all B.S.C.s on $\mathcal{D}$

Th$^{\underline{m}}$ [Bridgeland '07] $\operatorname{Stab}(\mathcal{D})$ is a complex manifold.

FROM NOW ON: X: smooth projective variety / $\mathbb{C}$

Q1 How can we relate X and $\operatorname{Stab}(X) := \operatorname{Stab}(D^b(X))$

Def$^{\underline{n}}$ $\sigma \in \operatorname{Stab}(X)$ is **geometric** if $\forall x \in X$, $\underbrace{\mathcal{O}_x}_{\text{skyscraper sheaf}}$ is σ-stable.

$$\left[\begin{array}{ccc} X & \hookrightarrow & M^{st}_\sigma(X) \\ x & \longmapsto & \mathcal{O}_x \end{array}\right]$$

§3 Geometric Stability Conditions

[**Def$^{\underline{n}}$** $\sigma \in \mathrm{Stab}(X)$ is **geometric** if $\forall x \in X$, $\mathcal{O}_x$ is σ-stable.]

What is known?

- $\dim X = 1$: $\mathrm{Stab}^{\mathrm{Geo}}(X) \simeq \mathbb{C} \times \mathbb{H}$
- $\dim X = 2$: General construction
- $\dim X = 3$: Some examples
- $\dim X \geq 4$: ???

Th$^{\underline{m}}$ [Lie Fu–Chunyi Li–Xiaolei Zhao '21]

X has finite **Albanese morphism** $\Longrightarrow$ $\mathrm{Stab}(X) = \mathrm{Stab}^{\mathrm{Geo}}(X)$

$$\boxed{\begin{tikzcd} X \arrow[r, "\mathrm{alb}_X"] \arrow[dr, "\forall f"'] & \mathrm{Alb}(X) \arrow[d, dashed, "\exists !"] \\ & A \end{tikzcd}}$$

Q2 alb_X not finite

?? $\Downarrow$??

$\exists$ non-geometric B.S.C.s?

§4: Free Abelian Quotients

(Q2) alb_X not finite

$$\overset{??}{\Downarrow}\ ??$$

$\exists$ non-geos?

IDEA:

$$\begin{tikzcd} G \arrow[r, phantom, "\overset{\text{free}}{\curvearrowright}"] & X \arrow[d, "\pi"] \\ & S := X/G \end{tikzcd}$$

finite abelian

F.A.Q. $\rightarrow$ $S := X/G$

§4: Free Abelian Quotients

Q2 alb_X not finite $\overset{??}{\Longrightarrow}$ $\exists$ non-geos?

IDEA:

$$\begin{tikzcd}
\underset{\text{finite abelian}}{G} \arrow[r, phantom, "\overset{\text{free}}{\curvearrowright}"] & X \arrow[r, "\text{finite}"] \arrow[d, "\pi"'] & \mathrm{Alb}(X) \\
\text{F.A.Q.} \arrow[r] & \boxed{S := X/G} \arrow[r, "\text{not finite}"] & \mathrm{Alb}(S)
\end{tikzcd}$$

§4: Free Abelian Quotients

Q2 alb_X not finite

?? $\Downarrow$??

$\exists$ non-geos?

IDEA: G finite abelian, $G \overset{\text{free}}{\curvearrowright} X$

$$\begin{tikzcd} X \arrow[r, "\text{finite}"] \arrow[d, "\pi"'] & \mathrm{Alb}(X) \\ S := X/G \arrow[r, "\text{not finite}"] & \mathrm{Alb}(S) \end{tikzcd}$$

F.A.Q. $\to$ $S := X/G$

Q: Does this happen?

A: Yes e.g. $\left\{ \begin{array}{l} \text{Beauville-type surfaces} \\ \text{Bielliptic surfaces} \end{array} \right.$

Thm [D.]
- X surface with alb_X finite
- $S = X/G$ F.A.Q.

Then $\mathrm{Stab}^{\mathrm{Geo}}(S) \simeq (\mathrm{Stab}(X))^G$ is a connected component of $\mathrm{Stab}(S)$.